

BZ Derivations of the Lie Algebra of 4×4 Antisymmetric Matrices over a Commutative Ring

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Abstract: Let R be a 2-torsion free commutative ring with identity. The authors denote by $L_4(R)$ the Lie algebra consisting of all 4×4 antisymmetric matrices over R . This paper gives the definition of extreme BZ derivations of $L_4(R)$ and the decomposition of an arbitrary BZ derivation and a sufficient and necessary condition for a BZ derivation to become the inner derivation.

1. Foreword

Let R be a commutative ring of 2-torsion free with 1, $L_4(R)$ is a set of antisymmetric matrices of order 4 on R . Definition

$$[A, B] = AB - BA, \forall A, B \in L_4(R)$$

Then $L_4(R)$ about $[A, B]$ operations form lie algebras, It is called a lie algebra of antisymmetric matrices of order 4 on R . In this paper, we main study the BZ derivations of the Lie algebra of 4×4 antisymmetric matrices. At present, many scholars have studied derivation and BZ derivation of matrix lie algebra. The derivation of parabolic subalgebras of general linear lie algebras $gl(n, R)$ is studied in [1]. In [2], the authors give the BZ derivation and its decomposition of strictly upper triangular matrix lie algebras. In [3], Li Qinghua gives the BZ derivation and its decomposition of upper triangular matrix lie algebra. In [4], Zhao Yu, Jin Ying, Kang Zhaomin, Li Dong and other scholars solved the BZ derivation and its decomposition of third-order antisymmetric matrix lie algebras over commutative rings, which can be decomposed into diagonal BZ derivations, extreme BZ derivations and central BZ derivations. In [5], Wang Ying and Guo Wenjie solved the BZ derivation and its decomposition of antisymmetric matrix algebra of order $n(n > 4)$ over 2-torsion free commutative ring. It can be decomposed into the sum of inner derivation and constant BZ derivation. In this paper, we give the BZ derivation and its decomposition of the Lie algebra of 4×4 antisymmetric matrices over a 2-torsion free commutative ring with 1. So far, the decomposition of BZ derivations of lie algebras of antisymmetric matrices over commutative rings is completely solved.

2. Basic Knowledge and Main Conclusions

Definition 2.1^[1] Let $\rho: L_4(R) \rightarrow L_4(R)$ be a linear transformation, If for arbitrary $A, B \in L_4(R)$,

$$\rho[A, B] = [\rho(A), B] + [A, \rho(B)]$$

Then ρ is called a derivation on $L_4(R)$.

Definition 2.2^[2] If a linear transformation $\phi: L_4(R) \rightarrow L_4(R)$ satisfies any $\forall A, B \in L_4(R)$, if $[A, B] = 0$, then $[\phi(A), B] + [A, \phi(B)] = 0$, Then ϕ is called a BZ derivation on $L_4(R)$.

Note It is easy to verify that the derivation must be a BZ derivation, but vice versa.

Set up $A_{ij} = E_{ij} - E_{ji}$ ($i < j$), Where (i, j) position of E_{ij} table is 1, square of order 4 with rest position 0. Easy to know

$A_{ij}, 1 \leq i < j \leq 4$ a set of bases for $L_4(R)$. for $\forall A \in L_4(R)$, let $A = \sum_{1 \leq i < j \leq 4} a_{ij} A_{ij}$, $a_{ij} \in R$. If we determine the image on the base, then we can give the image on the top. If we determine the image of ϕ on the base A_{ij} ($1 \leq i < j \leq 4$), then we can give the image of ϕ on $\forall A \in L_4(R)$.

Definition 2.3 For $\forall \alpha \in R, \forall A = \sum_{1 \leq i < j \leq 4} a_{ij} A_{ij} \in L_4(R)$, where $a_{ij} \in R$. defines linear changes:

$$\begin{aligned} \theta_{st}^{ij} : L_4(R) &\rightarrow L_4(R), \\ A &\mapsto \alpha(a_{ij} A_{st} + (-1)^{i+j+s+t} a_{ij} A_{st}^{-1}), \end{aligned}$$

Where $\overline{\{ij\}} = \{i'j'\}, \{i'j'\} \cap \{ij\} = \emptyset, i, j, i', j' \in \{1, 2, 3, 4\}$

Exchange of evidence, θ_{st}^{ij} is the BZ derivation on $L_4(R)$, which is called the extreme BZ derivation.

In this paper, the following theorems are given for the BZ derivation of the lie algebras of the fourth order antisymmetric matrices over a 2-torsion free commutative ring with 1:

Let ϕ be the BZ derivation on $L_4(R)$, then ϕ can be expressed as the sum of extreme BZ derivations, that is

$$\phi = \sum_{s < t, j=2,3,4} a_{st}^{1j} \theta_{st}^{1j}, a_{st}^{1j} \in R$$

3. Theorem Proving

Lemma 3.1 Let ϕ be the BZ derivation on $L_4(R)$, and $\phi(A_{ij}) = \sum a_{st}^{ij} A_{st}, 1 \leq i < j \leq 4$, then

$$\begin{aligned} \phi(A_{12}) &= a_{12}^{12} A_{12} + a_{13}^{12} A_{13} + a_{14}^{12} A_{14} + a_{23}^{12} A_{23} + a_{24}^{12} A_{24} + a_{34}^{12} A_{34}, \\ \phi(A_{34}) &= a_{34}^{12} A_{12} - a_{24}^{12} A_{13} + a_{23}^{12} A_{14} + a_{14}^{12} A_{23} - a_{13}^{12} A_{24} + a_{12}^{12} A_{34}. \end{aligned}$$

Prove $[\phi(A_{12}), A_{34}] + [A_{12}, \phi(A_{34})] = 0$ can be obtained from $[A_{12}, A_{34}] = 0$, by comparing the coefficients of $A_{13}, A_{14}, A_{23}, A_{24}$ respectively, we can obtain

$$a_{13}^{12} + a_{24}^{34} = 0, a_{14}^{12} - a_{23}^{34} = 0, a_{23}^{12} - a_{14}^{34} = 0, a_{24}^{12} + a_{13}^{34} = 0$$

Again by $[A_{12} + A_{23}, A_{34} + A_{14}] = 0$, we know

$$[\phi(A_{12}) + \phi(A_{23}), A_{34} + A_{14}] + [A_{12} + A_{23}, \phi(A_{34}) + \phi(A_{14})] = 0,$$

By comparison coefficient, we can know

$$-a_{12}^{12} - a_{14}^{14} + a_{23}^{23} + a_{34}^{34} = 0 \quad (1)$$

$$a_{34}^{12} + a_{23}^{14} - a_{14}^{23} - a_{12}^{34} = 0 \quad (2)$$

For the same reasons, because of $[A_{12} + A_{14}, A_{34} + A_{23}] = 0$, therefore

$$a_{12}^{12} - a_{14}^{14} + a_{23}^{23} - a_{34}^{34} = 0 \quad (3)$$

$$-a_{34}^{12} + a_{23}^{14} - a_{14}^{23} + a_{12}^{34} = 0 \quad (4)$$

From (1)+(3), we get $2a_{23}^{23} - 2a_{14}^{14} = 0$

From (2)+(4), we get $2a_{23}^{14} - 2a_{14}^{23} = 0$

Therefore $a_{12}^{12} = a_{34}^{34}, a_{34}^{12} = a_{12}^{34}, a_{23}^{23} = a_{14}^{14}, a_{23}^{14} = a_{14}^{23}$.

From the above equation, we can get the result of Lemma 3.1.

Lemma 3.2 Let ϕ be the BZ derivation on $L_4(R)$, and $\phi(A_{ij}) = \sum a_{st}^{ij} A_{st}, 1 \leq i < j \leq 4$, then

$$\begin{aligned} \phi(A_{13}) &= a_{12}^{13} A_{12} + a_{13}^{13} A_{13} + a_{14}^{13} A_{14} + a_{23}^{13} A_{23} + a_{24}^{13} A_{24} + a_{34}^{13} A_{34}, \\ \phi(A_{24}) &= -a_{34}^{13} A_{12} + a_{24}^{13} A_{13} - a_{23}^{13} A_{14} - a_{14}^{13} A_{23} + a_{13}^{13} A_{24} - a_{12}^{13} A_{34}. \end{aligned}$$

Prove $[\phi(A_{13}), A_{24}] + [A_{13}, \phi(A_{24})] = 0$ can be obtained from $[A_{13}, A_{24}] = 0$, by comparing the coefficients of $A_{12}, A_{14}, A_{23}, A_{34}$ respectively, we can obtain

$$a_{12}^{13} + a_{34}^{24} = 0, a_{14}^{13} + a_{23}^{24} = 0, a_{23}^{13} + a_{14}^{24} = 0, a_{34}^{13} + a_{12}^{24} = 0$$

Furthermore, from $[A_{12} + A_{13}, A_{34} - A_{24}] = 0$ and Lemma 3.1, we can know

$$a_{13}^{13} = a_{24}^{24}, a_{24}^{13} = a_{13}^{24}.$$

By summing up the above equations, we can get the conclusion of Lemma 3.2.

Lemma 3.3 Let ϕ be the BZ derivation on $L_4(R)$, and $\phi(A_{ij}) = \sum a_{st}^{ij} A_{st}, 1 \leq i < j \leq 4$, then

$$\begin{aligned} \phi(A_{14}) &= a_{12}^{14} A_{12} + a_{13}^{14} A_{13} + a_{14}^{14} A_{14} + a_{23}^{14} A_{23} + a_{24}^{14} A_{24} + a_{34}^{14} A_{34}, \\ \phi(A_{23}) &= a_{34}^{14} A_{12} - a_{24}^{14} A_{13} + a_{23}^{14} A_{14} + a_{14}^{14} A_{23} - a_{13}^{14} A_{24} + a_{12}^{14} A_{34}. \end{aligned}$$

Prove $[\phi(A_{14}), A_{23}] + [A_{14}, \phi(A_{23})] = 0$ can be obtained from $[A_{14}, A_{23}] = 0$, by comparing the coefficients of $A_{12}, A_{13}, A_{24}, A_{34}$ respectively, we can obtain

$$a_{12}^{14} - a_{34}^{23} = 0, a_{13}^{14} + a_{24}^{23} = 0, a_{24}^{14} + a_{13}^{23} = 0, a_{34}^{14} - a_{12}^{23} = 0$$

Also known by Lemma 3.1, $a_{23}^{23} = a_{14}^{14}, a_{23}^{14} = a_{14}^{23}$.

By summing up the above equations, we can get the conclusion of Lemma 3.3.

Theorem 3.1 Let ϕ be the BZ derivation on $L_4(R)$, then

$$\phi = \sum_{s < t, j=2,3,4} a_{st}^{1j} \theta_{st}^{1j}.$$

Prove All we have to prove is

$$\phi(A_{mm}) = \sum_{s < t, j=2,3,4} a_{st}^{1j} \theta_{st}^{1j}(A_{mm}), 1 \leq m < n \leq 4.$$

Here, we only prove

$$\phi(A_{12}) = \sum_{s < t, j=2,3,4} a_{st}^{1j} \theta_{st}^{1j}(A_{12}) \quad (5)$$

The rest of the equations can be obtained by the same principle.

$$\sum_{s<t, j=2,3,4} a_{st}^{1j} \theta_{st}^{1j} (A_{12}) = (a_{12}^{12} \theta_{12}^{12} + a_{34}^{12} \theta_{34}^{12} + a_{13}^{12} \theta_{13}^{12} + a_{24}^{12} \theta_{24}^{12} + a_{14}^{12} \theta_{14}^{12} + a_{23}^{12} \theta_{23}^{12})(A_{12})$$

$$= a_{12}^{12} A_{12} + a_{13}^{12} A_{13} + a_{14}^{12} A_{14} + a_{23}^{12} A_{23} + a_{24}^{12} A_{24} + a_{34}^{12} A_{34}$$

The proof of equation (5) is obtained from Lemma 3.1. So the theorem holds true.

We know $\phi = ad(-a_{23}^{13} A_{12} + a_{23}^{12} A_{13} + a_{24}^{12} A_{14} - a_{13}^{12} A_{23} - a_{14}^{12} A_{24} - a_{14}^{13} A_{34})$

$$+ a_{12}^{12} \theta_{12}^{12} + a_{13}^{13} \theta_{13}^{13} + a_{14}^{14} \theta_{14}^{14} + a_{34}^{12} \theta_{34}^{12} + a_{24}^{13} \theta_{24}^{13} + a_{23}^{14} \theta_{23}^{14}$$

$$+ (a_{13}^{12} + a_{12}^{13}) \theta_{12}^{13} + (a_{34}^{13} - a_{24}^{12}) \theta_{34}^{13} + (a_{14}^{12} + a_{12}^{14}) \theta_{12}^{14}$$

$$+ (a_{24}^{14} - a_{23}^{13}) \theta_{24}^{14} + (a_{14}^{13} + a_{13}^{14}) \theta_{14}^{13} + (a_{23}^{12} + a_{34}^{14}) \theta_{34}^{14}$$

Theorem 3.2 The BZ derivation on $L_4(R)$ is an inner derivation if and only if

$$a_{12}^{12} = a_{13}^{13} = a_{14}^{14} = a_{34}^{12} = a_{24}^{13} = a_{23}^{14} = 0,$$

$$a_{13}^{12} = -a_{12}^{13}, a_{14}^{12} = -a_{12}^{14}, a_{14}^{13} = -a_{13}^{14}, a_{23}^{12} = -a_{34}^{14}, a_{24}^{14} = a_{34}^{13}, a_{23}^{13} = a_{24}^{14}$$

Prove Necessity clearly, only proves adequacy. To prove

$$\phi = ad(-a_{23}^{13} A_{12} + a_{23}^{12} A_{13} + a_{24}^{12} A_{14} - a_{13}^{12} A_{23} - a_{14}^{12} A_{24} - a_{14}^{13} A_{34})$$

Just prove

$$\phi(A_{mm}) = ad(-a_{23}^{13} A_{12} + a_{23}^{12} A_{13} + a_{24}^{12} A_{14} - a_{13}^{12} A_{23} - a_{14}^{12} A_{24} - a_{14}^{13} A_{34})(A_{mm}), m < n.$$

Here, we only prove

$$\phi(A_{12}) = ad(-a_{23}^{13} A_{12} + a_{23}^{12} A_{13} + a_{24}^{12} A_{14} - a_{13}^{12} A_{23} - a_{14}^{12} A_{24} - a_{14}^{13} A_{34})(A_{12}) \quad (6)$$

The rest of the equations can be obtained by the same principle.

The right side of the equation is $a_{13}^{12} A_{13} + a_{24}^{12} A_{14} + a_{23}^{12} A_{23} + a_{24}^{12} A_{24}$.

Lemma 3.1 and above conditions know that equation (6) holds.

Thus the conclusion is obtained.

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